General Quantum Fidelity Susceptibilities for the $J_1 - J_2$ Chain

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We study slightly generalized quantum fidelity susceptibilities where the differential change in the fidelity is measured with respect to a different term than the one used for driving the system towards a quantum phase transition. As a model system we use the spin-1/2 $J_1 - J_2$ antiferromagnetic Heisenberg chain. For this model, we study three fidelity susceptibilities, χ_ρ, χ_D and $\chi_{\rm AF}$, which are related to the spin stiffness, the dimer order and antiferromagnetic order, respectively. All these ground-state fidelity susceptibilities are sensitive to the phase diagram of the $J_1 - J_2$ model. We show that they all can accurately identify a quantum critical point in this model occurring at $J_2^c \sim 0.241 J_1$ between a gapless Heisenberg phase for $J_2 < J_2^c$ and a dimerized phase for $J_2 > J_2^c$. This phase transition, in the Berezinskii-Kosterlitz-Thouless universality class, is controlled by a marginal operator and is therefore particularly difficult to observe.

I. INTRODUCTION

The study of quantum phase transitions, especially in one and two dimensions, is a topic of considerable and ongoing interest.[1] Recently the utility of a concept with its origin in quantum information, the quantum fidelity and the related fidelity susceptibility, was demonstrated for the study of quantum phase transitions (QPT).[2–5] It has since then been successfully applied to a great number of systems.[6–11] In particular, it has been applied to the $J_1 - J_2$ model that we consider here.[12]. For a recent review of the fidelity approach to quantum phase transitions, see Ref. 13. Most of these studies consider the case where the system undergoes a quantum phase transition as a coupling λ is varied. The quantum fidelity and fidelity susceptibility is then defined with respect to the same parameter. Apart from a few studies,[14–16] relatively little attention has been given to the case where the quantum fidelity and susceptibility are defined with respect to a coupling different than λ. Here we consider this case in detail for the $J_1 - J_2$ model and show that, if appropriately defined, these general fidelity susceptibilities may yield considerable information about quantum phase transitions occurring in the system and can be very useful in probing for a non-zero order parameter.

Without loss of generality, the Hamiltonian of any many-body system can be written as

$$H(\lambda) = H_0 + \lambda H_\lambda, \tag{1}$$

where λ is a variable which typically parametrizes an interaction and exhibits a phase transition at some critical value λ_c. In this form H_λ is then recognized as a term that *drives* the phase transition.[5] Using the eigenvectors of this Hamiltonian the ground-state (differential) fidelity can then be written as:

$$F(\lambda) = \left|\langle \Psi_0(\lambda)|\Psi_0(\lambda+\delta\lambda)\rangle\right|. \tag{2}$$

A series expansion of the GS fidelity in $\delta\lambda$ yields

$$F(\lambda) = 1 - \frac{(\delta\lambda)^2}{2}\frac{\partial^2 F}{\partial\lambda^2} + \ldots \tag{3}$$

where $\partial_\lambda^2 F \equiv \chi_\lambda$ is called the *fidelity susceptibility*. For a discussion of sign conventions and a more complete derivation see the topical review by Gu, Ref. 13. If the higher-order terms are taken to be negligibly small then the fidelity susceptibility is defined as:

$$\chi_\lambda(\lambda) = \frac{2(1-F(\lambda))}{(\delta\lambda)^2} \tag{4}$$

The scaling of χ_λ at a quantum critical point, λ_c, is often of considerable interest and has been studied in detail and previous studies[10,11,14,15,17] have shown that

$$\chi_\lambda \sim L^{2/\nu}, \quad \chi_\lambda/N \sim L^{2/\nu-d}, \tag{5}$$

with $N = L^d$ the number of sites in the system. An easy way to re-derive this result is by envoking finite-size scaling. Since $1-F$ obviously is *dimensionless* it follows from Eq. (4) that the appropriate finite-size scaling form for χ_λ is

$$\chi_\lambda \sim (\delta\lambda)^{-2} f(L/\xi). \tag{6}$$

If we now consider the case where the parameter λ drives the transition we may at the critical point λ_c identify $\delta\lambda$ with $\lambda - \lambda_c$. It follows that $\xi \sim (\delta\lambda)^{-\nu}$. As usual, we can then replace $f(L/\xi)$ by an equivalent function $\tilde{f}(L^{1/\nu}\delta\lambda)$. The requirement that χ_λ remains finite for a finite system when $\delta\lambda \to 0$ then implies that to leading order $\tilde{f}(x) \sim x^2 \sim L^{2/\nu}(\delta\lambda)^2$, from which Eq. (5) follows.

Here we shall consider a slightly more general case where the term driving the quantum phase transition is not the same as the one with respect to which the fidelity and fidelity susceptibility are defined. That is, one considers:

$$H(\lambda,\delta) = H_1 + \delta H_I, \quad H_1 = H_0 + \lambda H_\lambda. \tag{7}$$

The fidelity and the related susceptibility is then defined as

$$F(\lambda,\delta) = \left|\langle \Psi_0(\lambda,0)|\Psi_0(\lambda,\delta)\rangle\right|, \tag{8}$$

$$\chi_\delta(\lambda) = \frac{2(1-F(\lambda,\delta))}{\delta^2} \tag{9}$$

The scaling of χ_δ at λ_c for this more general case was derived by Venuti *et al.*[15] where it was shown that:

$$\chi_\delta \sim L^{2d+2z-2\Delta_v}, \quad \chi_\delta/N \sim L^{d+2z-2\Delta_v}. \tag{10}$$

Here, z is the dynamical exponent, d the dimensionality and Δ_v the scaling dimension of the perturbation H_I. In all cases that we consider here $z = d = 1$. We note that Eq. (10) assumes $[H_1, H_I] \neq 0$, if H_I commutes with H_1 then $F = 1$ and $\chi_\delta = 0$. The case where H_λ and H_I coincide is a special case of Eq. (10) for which $\Delta_V = d + z - 1/\nu$ and one recovers Eq. (5).

A particular appealing feature of Eq. (5) is that when $2/\nu > d$, χ_λ/N will diverge at λ_c and the fidelity susceptibility can then be used to locate the λ_c *without* any need for knowing the order parameter. Secondly, it can be shown[5,14] that the fidelity susceptibility can be expressed as the zero-frequency *derivative* of the dynamical correlation function of H_I, making it a very sensitive probe of the quantum phase transition.[18] On the other hand, if a phase transition is expected one might then use the fidelity susceptibility as a very sensitive probe of the order parameter through a suitably defined H_δ in Eq. (7). This is the approach we shall take here using the $J_1 - J_2$ spin chain as our model system.

The spin-1/2 Heisenberg $J_1 - J_2$ chain is a very well studied model. The Hamiltonian is:

$$H = \sum_i S_i \cdot S_{i+1} + J_2 \sum_i S_i \cdot S_{i+2} \tag{11}$$

where J_2 is understood to be the ratio of the next-nearest neighbor exchange parameter over the nearest neighbor exchange parameter ($J_2 = J_2'/J_1'$). This model is known to have a quantum phase transition of the Berezinskii-Kosterlitz-Thouless (BKT) universality class occurring at J_2^c between a gapless 'Heisenberg' (Luttinger liquid) phase for $J_2 < J_2^c$ and a dimerized phase with a two-fold degenerate ground-state for $J_2 > J_2^c$. Field theory[19,20], exact diagonalization[21,22] and DMRG[23,24], have yielded very accurate estimates of the Luttinger Liquid-Dimer phase transition, the most accurate of these being due to Eggert[22] which yielded a value of $J_2^c = 0.241167$. Previous studies by Chen *et al.*[12] of this model using the fidelity approach used the same term for the driving and perturbing part of the Hamiltonian as in Eq. (1) with the correspondence $H_0 = \sum_i S_i \cdot S_{i+1}$, $H_\lambda = \sum_i S_i \cdot S_{i+2}$, $\lambda = J_2$.[12] . Chen *et al.* demonstrated that, though no useful information about the Luttinger Liquid-Dimer phase transition could be obtained directly from the *ground-state* fidelity (and similarly the fidelity susceptibility), a clear signature of the phase transition was present in the fidelity of the *first excited* state.[12] Sometimes this is taken as an indication that ground-state fidelity susceptibilities are not useful for locating a quantum phase transition in the BKT universality class. Here we show that more general ground-state fidelity susceptibilities indeed can locate this transition.

Specifically, we will study three fidelity susceptibilities, χ_ρ, χ_D and χ_{AF}, which are coupled to the spin stiffness, a staggered interaction term and a staggered field term, respectively. In section II we present our results for χ_ρ while section III is focused on χ_D and section IV on χ_{AF}.

II. THE SPIN STIFFNESS FIDELITY SUSCEPTIBILITY, χ_ρ

We begin by considering the $J_1 - J_2$ model with $J_2 = 0$ but with an anisotropy term Δ, what is usually called the XXZ model:

$$H_{\mathrm{XXZ}} = \sum_i [\Delta S_i^z S_{i+1}^z + \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)]. \tag{12}$$

The Heisenberg phase of this model, occurring for $\Delta \in [-1, 1]$, is characterized by a non-zero spin stiffness[25,26] defined as:

$$\rho(L) = \left. \frac{\partial^2 e(\phi)}{\partial \delta^2} \right|_{\phi=0}. \tag{13}$$

Here, $e(\phi)$ is the ground-state energy per spin of the model where a twist of ϕ is applied at every bond:

$$H_{\mathrm{XXZ}}(\Delta, \phi) = \sum_i [\Delta S_i^z S_{i+1}^z + \frac{1}{2}(S_i^+ S_{i+1}^- e^{i\phi} + S_i^- S_{i+1}^+ e^{-i\phi})]. \tag{14}$$

The spin stiffness can be calculated exactly for the XXZ model for finite L using the Bethe ansatz,[27] and exact expressions in the thermodynamic limit are available. [25,26] Interestingly the usual fidelity susceptibility with respect to Δ can also be calculated exactly.[28,29]

Since the non-zero spin stiffness defines the gapless Heisenberg phase it is therefore of interest to define a fidelity susceptibility associated with the stiffness. This can be done through the overlap of the ground-state with $\phi = 0$ and a non-zero ϕ. With $\Psi_0(\Delta, \phi)$ the ground-state of $H_{\mathrm{XXZ}}(\Delta, \phi)$ we can define the fidelity and fidelity susceptibility with respect to the twist in the limit where $\phi \to 0$:

$$F(\Delta, \phi) = |\langle \Psi_0(\Delta, 0) | \Psi_0(\Delta, \phi) \rangle| \,, \tag{15}$$

$$\chi_\rho(\Delta) = \frac{2(1 - F(\Delta, \phi))}{\phi^2}. \tag{16}$$

To calculate χ_ρ the ground-state of the unperturbed Hamiltonian was calculated through numerical exact diagonalization. The system was then perturbed by adding a twist of $e^{i\phi}$ at each bond and recalculating the ground-state. From the corresponding fidelity, χ_ρ was calculated using Eq. (16). Our results for χ_ρ/L versus Δ are shown in Fig. 1. For all data ϕ was taken to be 10^{-3} and periodic boundary conditions were assumed. In all cases it was verified that the finite value of ϕ used had no effect on the final results. The numerical diagonalizations were done using the Lanczos method as outlined by Lin *et al.*[30] Total S^z symmetry and parallel programming techniques

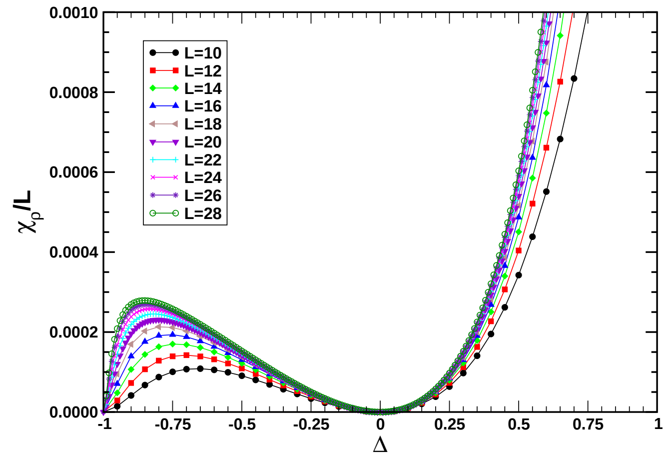

FIG. 1. (Color online.) $\frac{\chi_\rho}{L}$ vs. Δ: The spin stiffness fidelity susceptibility $(\chi_\rho(\Delta)/L)$ as a function of the z-anistropy Δ. At the $\Delta = 0$ point the spin-current operator $\mathcal{J}$ and kinetic energy $\mathcal{T}$ commute with the XXZ Hamiltonian and thus such a perturbation does not change the ground-state, and the fidelity is one. Thus, χ_ρ is zero at this point.

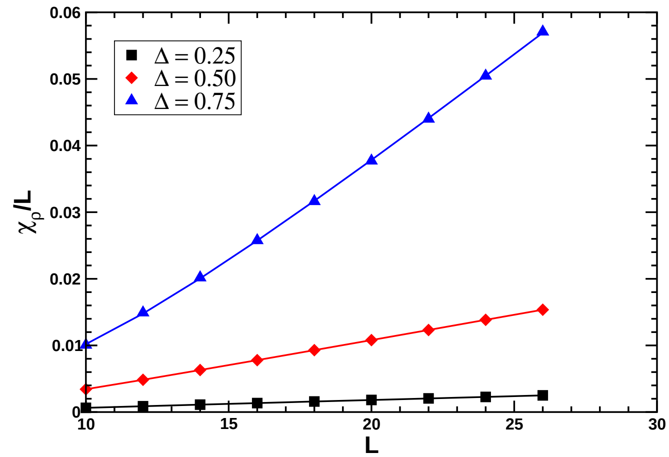

FIG. 2. (Color online.) χ_ρ vs. L (the XXZ model at different values of Δ): This graph shows the scaling of χ_ρ with system size for different values of the z-anisotropy Δ. The points represent numerical data and the lines represent fits to the scaling form predicted for the spin stiffness susceptibility $\chi_\rho/L = A_1 L + A_2 + A_3 L^{-1} + A_4 L^{3-8K}$. It can be seen that there is good agreement.

were employed to make computations feasible. Numerical errors are small and conservatively estimated to be on the order of 10^{-10} in the computed ground-state energies.

In order to understand the results in Fig. 1 in more detail we expand Eq. (14) for small ϕ:

$$H_{\mathrm{XXZ}}(\Delta,\phi) \sim H_{\mathrm{XXZ}}(\Delta) + \phi\mathcal{J} - \frac{\phi^2}{2}\mathcal{T} + \ldots, \quad (17)$$

$$\mathcal{J} = \frac{i}{2}\sum_i (S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+), \quad (18)$$

$$\mathcal{T} = \frac{1}{2}\sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+). \quad (19)$$

Here, $\mathcal{J}$ is the spin current and $\mathcal{T}$ a kinetic energy term. The first thing we note is that, when $\Delta = 0$ both $\mathcal{J}$ and $\mathcal{T}$ commute with $H_{\mathrm{XXZ}}(\Delta = 0)$. The ground-state wavefunction is therefore independent of ϕ (for small ϕ) and $\chi_\rho \equiv 0$. This can clearly be seen in Fig. 1.

In the continuum limit the spin current $\mathcal{J}$ can be expressed in an effective low energy field theory[31] with scaling dimension $\Delta_{\mathcal{J}} = 1$. However, we expect subleading corrections to arise from the presence of the operators $(\partial_x\Phi)^2$ with scaling dimension 2 and $\cos(\sqrt{16\pi K}\Phi)$ with scaling dimension $4K$. Here, K is given by $K = \pi/(2(\pi - \arccos(\Delta)))$. For $\Delta \neq 0$ both of these terms will be generated by the term $\mathcal{T}$ in Eq. (17).[15] With these scaling dimensions and with the use of Eq. (10) we then find:

$$\chi_\rho/L = A_1 L + A_2 + A_3 L^{-1} + A_4 L^{3-8K} \quad (20)$$

In Fig. 2 a fit to this form is shown for 3 different values of $\Delta = 0.25, 0.5$ and 0.75 in all cases do we observe an excellent agreement with the expected form with corrections arising from the last term L^{3-8K} being almost un-noticeable until Δ approaches 1. We would expect the sub-leading corrections L^{-1} and L^{3-8K} to be absent if the perturbative term is just $\phi\mathcal{J}$.

We now turn to a discussion of a definition of χ_ρ in the presence of a non-zero J_2 but restricting the discussion to the isotropic case $\Delta = 1$. In this case we define:

$$H(\phi) = \sum_i [S_i^z S_{i+1}^z + \frac{1}{2}(S_i^+ S_{i+1}^- e^{i\phi} + S_i^- S_{i+1}^+ e^{-i\phi})] + J_2 \sum_i [S_i^z S_{i+2}^z + \frac{1}{2}(S_i^+ S_{i+2}^- e^{i\phi} + S_i^- S_{i+2}^+ e^{-i\phi})]. \quad (21)$$

That is, we simply apply the twist ϕ at every bond of the Hamiltonian. As before we can expand:

$$H(\phi) \sim H(0) + \phi(\mathcal{J}_1 + \mathcal{J}_2) - \frac{\phi^2}{2}(\mathcal{T}_1 + \mathcal{T}_2) + \ldots, (22)$$

$$\mathcal{J}_1 = \frac{i}{2}\sum_i (S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+), \quad (23)$$

$$\mathcal{J}_2 = \frac{i}{2}\sum_i (S_i^+ S_{i+2}^- - S_i^- S_{i+2}^+), \quad (24)$$

$$\mathcal{T}_1 = \frac{1}{2}\sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+), \quad (25)$$

$$\mathcal{T}_2 = \frac{1}{2}\sum_i (S_i^+ S_{i+2}^- + S_i^- S_{i+2}^+). \quad (26)$$

Our results for χ_ρ/L versus J_2 using this definition are shown in Fig. 3 for a range of L from 10 to 32. In the region of the critical point at $J_2 = 0.241167$ the size dependence of χ_ρ/L vanishes yielding near scale invariance. How well this works close to J_2^c is shown in the inset of

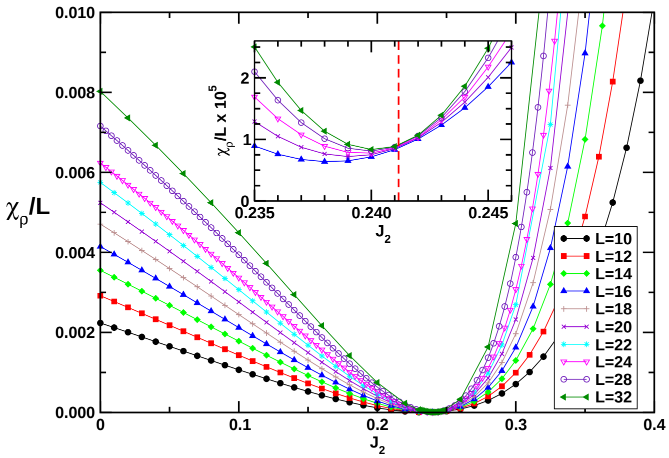

FIG. 3. (Color online.) $\frac{\chi_\rho}{L}$ vs. J_2 and Inset: The generalized spin stiffness susceptibility, χ_ρ as a function of the second nearest-neighbor exchange coupling J_2. The system acquires a clearly size invariant form in the vicinity of the critical point $J_2 \sim 0.24$ (as well as tending to a global minima). Inset shows the the minima for system sizes L=16,20,24,28,32 with J_2^c indicated as the vertical dashed line. A clear dependence of the J_2 value of χ_ρ/L minima on the system size can be seen.

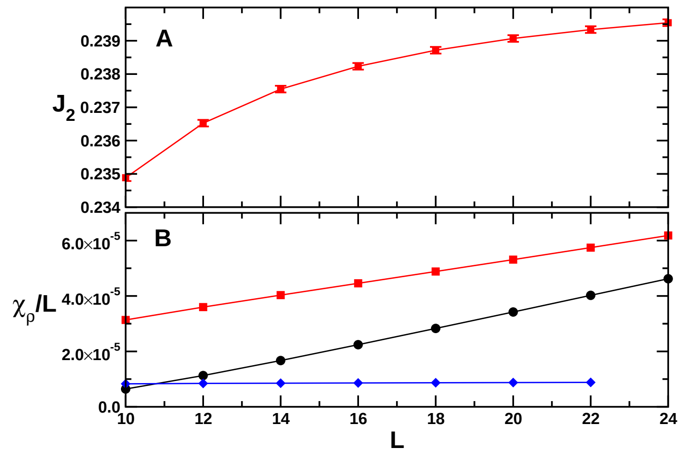

FIG. 4. (Color online.) A: The J_2 value of χ_ρ minima as a function of system size, as well as a (power law) line of best fit. As the system size tends towards infinity the power law best fit predicts a minima at $J_2 = 0.24077$ in good agreement with previously published results. B: Scaling of χ_ρ at $J_2 = 0.23$ (the highest, linear curve), $J_2 = 0.25$ (the second highest, linear curve) and at the critical point $J_2 = 0.241167$ (flat curve). The near constant scaling of $\frac{\chi_\rho}{L}$ at the critical point as well as non-constant scaling on either side of the critical point can clearly be seen.

Fig. 3. This alone can be taken to be a strong indication of χ_ρ/Ls sensitivity to the phase transition. In fact, this scale invariance works so well that one can locate the critical point to a high precision simply by verifying the scale invariance. This is illustrated in Fig. 4B where χ_ρ/L is plotted as a function of L for $J_2 = 0.23$, $J_2 = J_2^c$ and $J_2 = 0.25$. From the results in Fig. 4B the critical point J_2^c where χ_ρ/L becomes independent of L is immediately visible.

As can be seen in the inset of Fig. 3 χ_ρ/L reaches a minimum slightly prior to J_2^c. The J_2 value at which this minimum occurs has a clear system size dependence which can be fitted to a power-law and extrapolated to $L = \infty$ yielding a value of $J_{2c} = 0.24077$. Hence, the minimum coincides with J_2^c in the thermodynamic limit. This is shown in Fig. 4A. Comparison of this value with the accepted $J_2^c = 0.241167$ reveals impressive agreement. Another noteworthy feature of the results in Fig. 3 is that χ_ρ/L is *non-zero* at the critical point, J_2^c. This value is very small but we have verified in detail that numerically it is non-zero.

The scale invariance of χ_ρ/L is clearly induced by the disappearance[20] of the marginal operator $\cos(\sqrt{16\pi K}\Phi)$ at J_2^c. We expect that in the continuum limit the absence of this operator implies that the spin current commutes with the Hamiltonian resulting in χ_ρ being effectively zero at J_2^c. The observed non-zero value of χ_ρ/L would then arise from short-distance physics.

Note that, as mentioned previously, we take the spin stiffness to be represented by a twist on *every* bond, both first and second nearest neighbor and not merely on the boundary as is sometimes done. This choice is not just a matter of taste. Imposing a twist only on the boundary (usually) breaks the translational invariance of the ground-state and, through extension, effects the value and behavior of the fidelity itself. Another point of note is the use of a twist of only ϕ between next-nearest neighbors. Geometric intuition would suggest that a twist of 2ϕ should be applied between next-nearest neighbor bonds. However, for the small system sizes available for exact diagonlization it is found that a simple twist of ϕ on both bonds yields *significantly* better scaling.

III. THE DIMER FIDELITY SUSCEPTIBILITY, χ_D

We now turn to a discussion of a fidelity susceptibility associated with the dimer order present in the $J_1 - J_2$ model for $J_2 > J_2^c$. This susceptibility, which we call χ_D, is coupled to the order parameter of the dimerized phase by design. Usually in the fidelity approach to quantum phase transitions one considers the case where the ground-state is unique in the absence of the perturbation. This is not the case here, leading to a diverging χ_D/L in the dimerized phase even in the presence of a gap. Specifically, we consider a Hamiltonian of the form:

$$H = \sum_i [S_i \cdot S_{i+1} + J_2 S_i \cdot S_{i+2} + \delta h (-1)^i S_i \cdot S_{i+1}] \tag{27}$$

Thus, in correspondence with Eq. (7) we have $H_I = (-1)^i S_i \cdot S_{i+1}$ and we choose the driving coupling to be

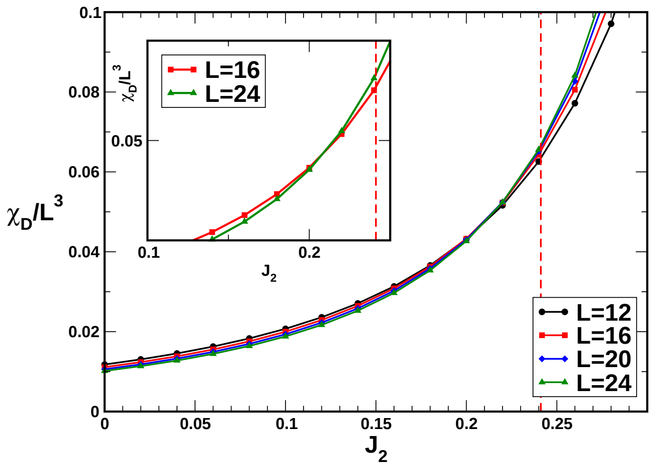

FIG. 5. (Color online.) $\frac{\chi_D}{L^3}$ vs. J_2. The generalized dimer fidelity susceptibility χ_D/L^3 as a function of the second nearest-neighbor exchange parameter J_2. A clear intersection of all curves can be seen in the vicinity of the proposed critical point at $J_2 \sim 0.2-0.25$. The inset explicitly shows the crossing of $L = 12$ and $L = 24$. The dashed vertical lines indicate J_2^c.

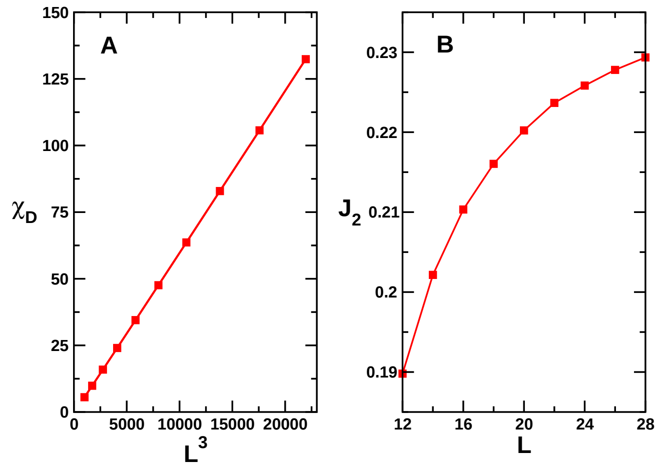

FIG. 6. (Color online.) A: Scaling of χ_D vs. L^3 at the critical point $J_2 = 0.241167$. A near perfectly linear scaling is observed. B: The J_2 value of the intersection of $\frac{\chi_D}{L^3}$ between systems of size L and $L+2$ plotted as a function of L. The curve can be fitted with a power-law line of best fit. The line of best fit is found to converge to $J_2 = 0.241$.

J_2. This perturbing Hamiltonian represents a *conjugate field* for the dimer phase. The scaling dimension of H_I is known[32], $\Delta_D = \frac{1}{2}$, and from Eq. (10) we therefore find:

$$\chi_D \sim L^{4-2\Delta_D} = L^3 \quad (\text{at } J_2^c) \tag{28}$$

Due to the presence of the marginal coupling we cannot expect this relation to hold for $J_2 < J_2^c$. However, the marginal coupling changes sign at J_2^c and is therefore absent at J_2^c where Eq. (28) should be exact.[20] For $J_2 < 0.241167$ it is known[32] that logarithmic corrections arising from the marginal coupling for the small system sizes considered here lead to an effective scaling dimension $\Delta_D > \frac{1}{2}$. At $J_2 = 0$ Affleck and Bonner[32] estimated $\Delta_D = 0.71$. Hence, using this results at $J_2 = 0$, we would expect that $\chi_D \sim L^{2.58}$ which we find is in good agreement with our results at $J_2 = 0$.

We now need to consider the case $J_2 > 0.241167$. At $J_2 = 1/2$ the model is exactly solvable[33] and the two dimerized ground-states are exactly degenerate even for finite L. For $J_2^c < J_2 < 1/2$ the system is gapped with a unique ground-state but with an exponentially low-lying excited state. In the thermodynamic limit the two-fold degeneracy of ground-state is recovered, corresponding to the degeneracy of the two dimerization patterns. From this it follows that χ_D is formally infinite at $J_2 = 0$ and as $L \to \infty$ for $J_2^c < J_2 < 1/2$ we expect χ_D to diverge exponentially with L. At J_2^c we expect χ_D to exactly scale as L^3 and for $J_2 < J_2^c$ we expect $\chi_D \sim L^{\alpha}_{\text{eff}}$ with $\alpha_{\text{eff}} < 3$. Hence, if χ_D/L^3 is plotted for different L we would expect the curves to cross at J_2^c. However, the crossing might be difficult to observe since it effectively arise from logarithmic corrections.

Our results for χ_D/L^3 are shown in Fig. 5, where a crossing of the curves are visible around $J_2 \sim 0.2-0.25$. As an illustration, the inset of Fig. 5 shows the crossing of $L = 12$ and $L = 24$. In order to obtain a more precise estimate of J_2^c the intersection of each curve and the curve corresponding to the next largest system were tabulated (L and $L+2$). These intersection points as a function of system size were then plotted Fig. 6A and found to obey a power-law of the form $a - bL^{-\alpha}$ with $\alpha \sim 1.8$ and $a = 0.241$. This estimate of the critical coupling is in good agreement with the value of $J_2^c = 0.241167$.[22]

To further verify the scaling of χ_D at J_2^c we show in Fig. 6B χ_D at J_2^c as a function of the cubed system size, L^3. The strong linear scaling is in contrast to the scaling a small distance away from the critical point (not shown) where the scaling was found to be distorted by logarithmic corrections.

IV. THE AF FIDELITY SUSCEPTIBILITY, χ_{AF}

Finally, we briefly discuss another fidelity susceptibility very analogous to χ_D. We consider a perturbing term in the form of a staggered field of the form $\sum_i (-1)^i S_i^z$ with an associated fidelity susceptibility, χ_{AF}. The scaling dimension of such a staggered field is $\Delta_{\text{AF}} = \frac{1}{2}$ and as for χ_D we therefore expect that $\chi_{\text{AF}} \sim L^3$ at J_2^c. However, in this case it is known[32] that the effective scaling dimension for $J_2 < J_2^c$ is *smaller* than $\frac{1}{2}$ resulting in $\chi_{\text{AF}} \sim L^{\alpha_{\text{eff}}}$ with $\alpha_{\text{eff}} > 3$ for $J_2 < J_2^c$. On the other had, in the dimerized phase χ_{AF} must clearly go to zero exponentially with L. Hence, if χ_{AF} is plotted for different L as a function of J_2 a crossing of the curves should occur.

Our results are shown in Fig. 7 where χ_{AF}/L^3 is plotted versus J_2 for a number of system sizes. It is clear

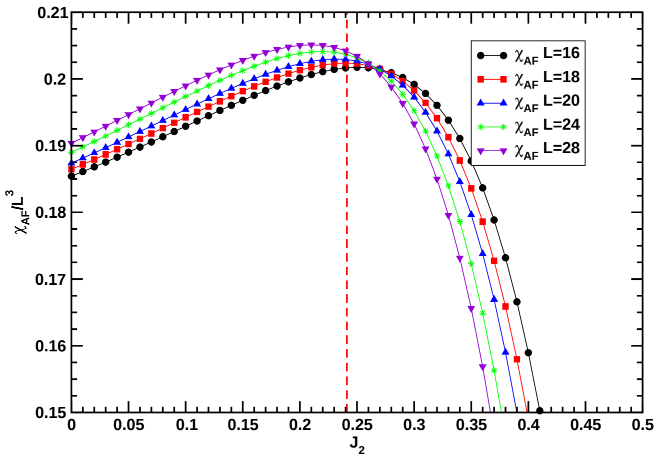

FIG. 7. (Color online.) $\chi_{\rm AF}/L^3$ versus J_2. $\chi_{\rm AF}$ is expected to approach zero exponentially with the system size for $J_2 > J_2^c$, to scale as L^3 at J_2^c and to scale as $L^{\alpha_{\rm eff}}$ with $\alpha_{\rm eff} > 3$ for $J_2 < J_2^c$. A crossing close to the critical point J_2^c (dashed vertical line) is then visible.

from these results that $\chi_{\rm AF}$ indeed goes to zero rapidly in the dimerized phase as one would expect. Close to J_2^c the scaling is close to L^3 where as for $J_2 < J_2^c$ it is faster than L^3. Hence, as can be seen in Fig. 7, a crossing occurs close to J_2^c.

V. CONCLUSION AND SUMMARY

In this paper we have demonstrated the potential benefits of extending the concept of a fidelity susceptibility beyond a simple perturbation of the same term that drives the quantum phase transition. By using the spin-1/2 Heisenberg spin chain as an example we first created a susceptibility which was directly coupled to the spin stiffness but of increased sensitivity. This fidelity susceptibility, which we labelled χ_ρ can be used to successfully estimate the transition point at $J_2 \sim 0.241$. Next we constructed another fidelity susceptibility, χ_D, this time coupled to the order parameter susceptibility of the dimer phase. Again, we were able to estimate the critical point at a value of 0.241. Finally, we discussed an anti ferromagnetic fidelity susceptibility that rapidly approaches zero in the dimerized phase but diverges in the Heisenberg phase. Although susceptibilities linked to these quantities appeared the most useful for the $J_1 - J_2$ model we considered here, it is possible to define many other fidelity susceptibilities that could provide valuable insights into the ordering occurring in the system being studied.

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